MK ML RL AB KS

NAME_______________

CLASS 12MTX___

CHERRYBROOK TECHNOLOGY HIGH SCHOOL

2019

YEAR 12

AP4

MATHEMATICS EXTENSION 1

Time allowed – 2 HOURS + 5 Minutes Reading Time

DIRECTIONS TO CANDIDATES:

SECTION I: Use the multiple-choice answer sheet provided.

SECTION II:

- Each question is to be commenced in a new booklet clearly marked Question 11, Question 12, etc. on the front.
- All necessary working should be shown in every question. Full marks may not be awarded for careless or badly arranged work.
- If you do not attempt a question, you must submit a blank booklet clearly indicating the question number, your name and class.
- All questions should be placed in order – multiple choice answer sheet followed by Questions 11 to 14.
- NESA approved calculators may be used.
- The NESA Reference Sheet is provided.

Section I

10 Marks

Attempt Questions 1 – 10.

Allow about 15 minutes for this section.

Use the multiple choice answer sheet for questions 1 – 10.

1. The acute angle between the lines $y = 2x$ and $y = 6x$ is θ. What is the value of $\tan\theta$?

(A) $\frac{4}{13}$

(B) $\frac{8}{11}$

(C) $\frac{1}{8}$

(D) $\frac{4}{9}$

2. What is the remainder when $P(x) = x^3 - 10x$ is divided by $x + 5$?

(A) -75

(B) 75

(C) $x^2 - 2x$

(D) $x^2 - 5\mathrm{x} + 15$

3. If $\cos 2\theta = \frac{1}{3}$ and $0 \leq \theta \leq \frac{\pi}{2}$, what is the value of $\tan\theta$?

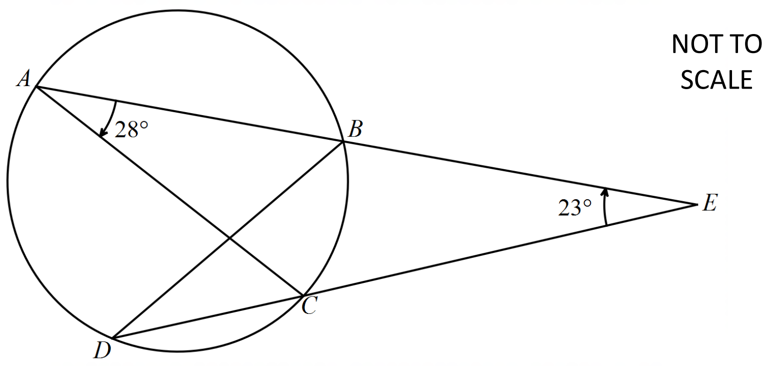
(A) $\frac{1}{2}$

(B) $\frac{1}{\sqrt{2}}$

(C) $\sqrt{2}$

(D) 2

4. In the diagram, ABC is a circle. D is on the circumference of the circle, with AB and DC produced to meet at E. $\angle BAC = 28°$ and $\angle DEA = 23°$.



What is the size of $\angle DBE$?

(A) 129°

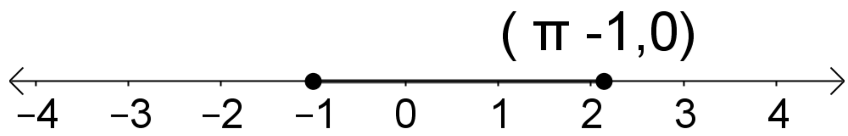
(B) 134°

(C) 152°

(D) 141°

5. Which diagram represents the domain of the function $f(x) = \cos^{-1}(x + 1)$?

(A)



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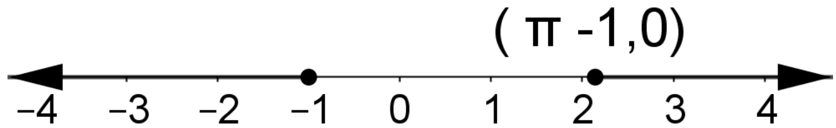

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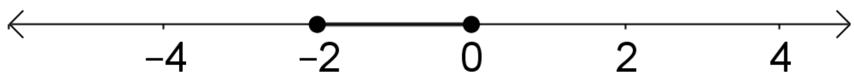

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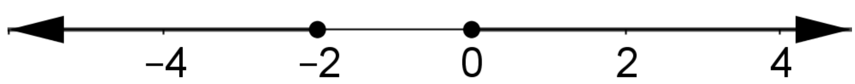


6. The displacement x of a particle at time t is given by $x = 4\cos 3t$.

What is the maximum speed of the particle?

(A) -12

(B) 12

(C) 6

(D) -6

7. A particle moves in a straight line. Its position at any time *t* is given by:

$$x = 4\sin 2t + 3\cos 2t$$

What is the acceleration in terms of *x*?

(A) $\ddot{x} = -4x$

(B) $\ddot{x} = -8x$

(C) $\ddot{x} = -16x^2$

(D) $\ddot{x} = -16\sin 2x - 12\cos 2x$

8. What is the value of $\lim\limits_{x\to 2}\frac{\sin(x-2)}{x^2+3x-10}$?

(A) undefined

(B) 0

(C) 7

(D) $\frac{1}{7}$

9. Which of the following is an expression for $\int \frac{x}{(2-x^2)^3}\,dx$?

Use the substitution $u = 2 - x^2$.

(A) $\int \frac{du}{u^3}$

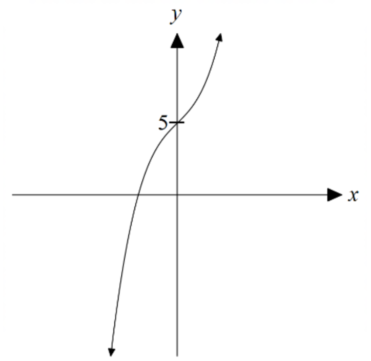
(B) $\int \frac{du}{-u^3}$

(C) $\int \frac{du}{2u^3}$

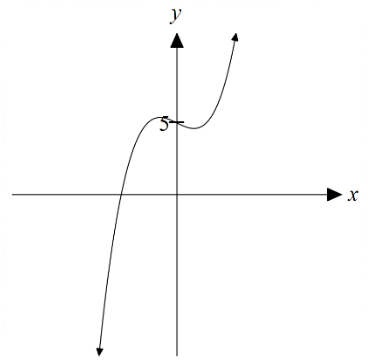
(D) $\int \frac{du}{-2u^3}$

10. Consider the polynomial $p(x) = ax^3 + cx + 5$ with a and c positive.
Which graph could represent of $p(x)$?

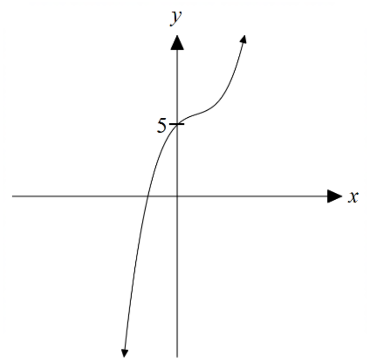
(A)



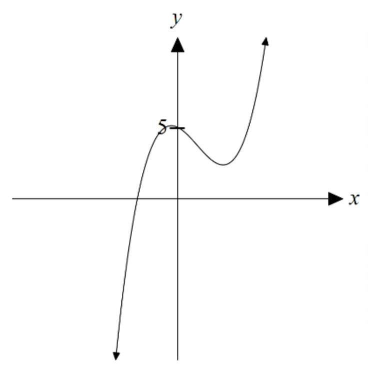
(B)



(C)



(D)



End of Section 1

Section II

60 Marks

Attempt Questions 11 – 14.

Allow about 1 hour and 45 minutes for this section.

Answer each question in a separate writing booklet.

Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 Marks) Start a new writing booklet. **Marks**

a) The point P internally divides the interval from $A(-2,\ 2)$ to $B(3,\ 5)$ into the ratio of 3:1. Find the *y*-coordinate of P. **1**

b) Solve $\frac{3x}{2x+3} + 1 \geq 0$. **3**

c) Differentiate $2x^2 \cos^{-1}(3x)$ **2**

d) Find $\int \cos^2 4x\, dx$ **2**

e) Use the substitution $u = e^{2x}$ to evaluate, **3**

$$\int_0^{\frac{1}{2}} \frac{e^{2x}}{e^{4x}+1}\, dx$$

giving your answer in exact form.

f) Find the constant term of the binomial expansion of $\left(2x - \frac{3}{x^2}\right)^9$? **2**

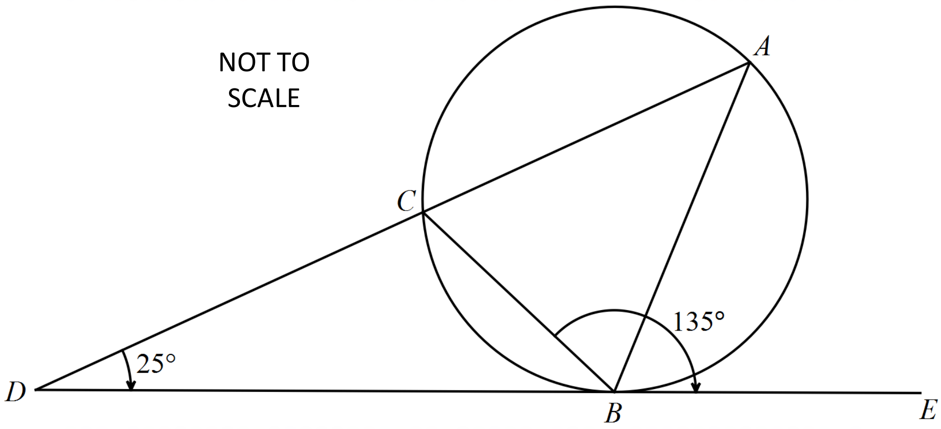
g) Use Newton's method to find a second approximation to the root of the function $f(x) = \sin(e^x)$. Take $x = 5$ as the first approximation. Answer correct to two decimal places. **2**

End of Question 11

Question 12 (15 Marks) Start a new writing booklet. **Marks**

a) The points A, B, and C lie on a circle. DBE is tangent at B, and ACD is a straight line. **2**

It is given that $\angle ADE = 25°$ and $\angle CBE = 135°$.



Find the size of $\angle ABC$, giving reasons.

b) Ice cream taken out of a freezer has a temperature of -18°C. It is placed in a room with a constant temperature of 25°C. After t minutes the temperature, T°C, of the ice cream can be given by

$$T = A - Be^{-0.025t},$$

where A and B are positive constants.

(i) Show that T is a solution to $\frac{dT}{dt} = -0.025(T - A)$ **1**

(ii) How long does it take for the ice cream to reach its melting point of -3°C? **3**

c) Let $f(x) = \dfrac{1}{\sqrt{2x+1}}$

(i) Find the domain and range of $f(x)$. **2**

(ii) Find an expression for the inverse function $f^{-1}(x)$. **2**

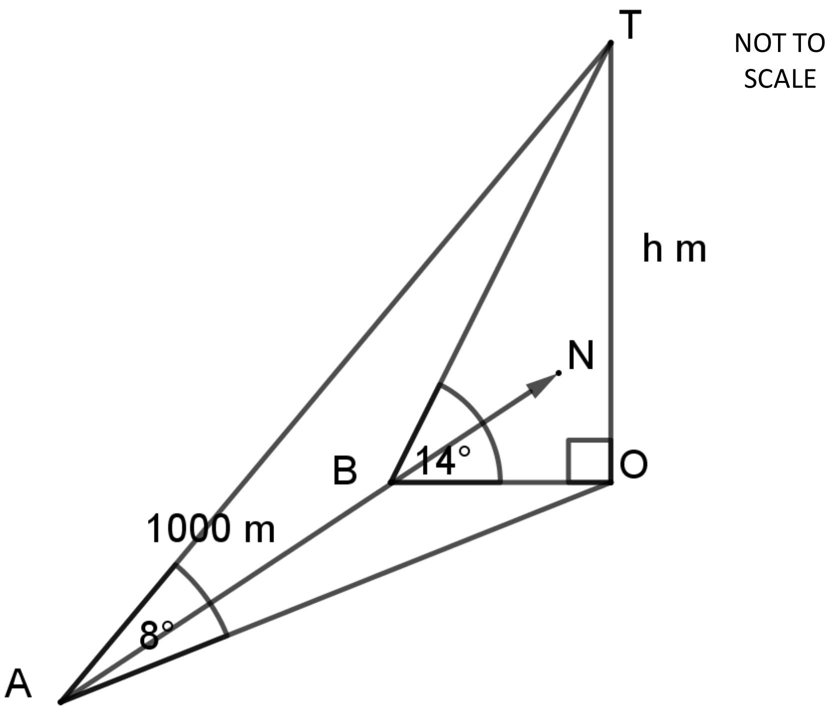
(iii) On the same set of axes, sketch the graphs $y = f(x)$ and $y = f^{-1}(x)$, clearly marking the x- or y-intercepts and any asymptotes which are not the x- or y-axes. **2**

Question 12 continued over page

d) A person is travelling in a car along a road from point A to point B, which is 1000 metres due north of Point A. Point B is due west of a tower OT, where T is the top of the tower. The point O is directly below T and on the same horizontal plane as the road. The height of the tower above O is h metres.

From point A the angle of elevation of the top of the tower is 8°.

From point B the angle of elevation of the top of the tower is 14°.



(i) Show that $OB = h \cot 14°$ **1**

(ii) Hence, find the value of h. **2**

End of Question 12

Question 13 (15 Marks) Start a new writing booklet. **Marks**

a) A particle is moving along the x-axis, such that its displacement at time t is x, and its velocity is v in units of metres and seconds. The particle is moving such that its velocity is given by $v^2 = -4x^2 - 8x + 12$.

(i) Show that the particle moves in simple harmonic motion with a period of π. **2**

(ii) What is the maximum speed of the particle? **1**

b) (i) Differentiate $x \tan^{-1} x - \frac{1}{2}\ln(1 + x^2)$. **2**

(ii) Hence, evaluate $\int_0^1 \tan^{-1} x \, dx$, giving your answer in exact form. **1**

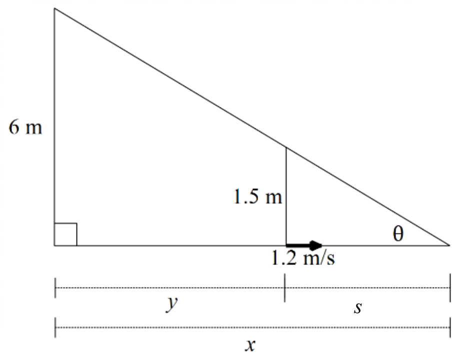
c) The function $f(x)$ is given by $f(x) = \sin^{-1} x + \cos^{-1} x$

(i) Find $f'(x)$ **1**

(ii) Hence sketch the graph of $y = \sin^{-1} x + \cos^{-1} x$ **1**

d) The velocity of a particle is given by $v = 3x + 7$ cm/s. If the particle is initially 1cm to the right of the origin, find the displacement as a function of time. **3**

e) A person who is 1.5 metres tall is walking away from a light positioned on a pole that is 6 metres tall. The person walks at a constant speed of 1.2 metres per second.



NOT TO SCALE

(i) Using similar triangles, or otherwise, show that the rate at which s, the length of the shadow, is increasing with respect to time is 0.4 m/s. **1**

(ii) By differentiating $s \tan\theta = 1.5$, or otherwise, find the rate at which the angle θ is changing with respect to time when the person is 5 metres from the base of the light pole. **3**

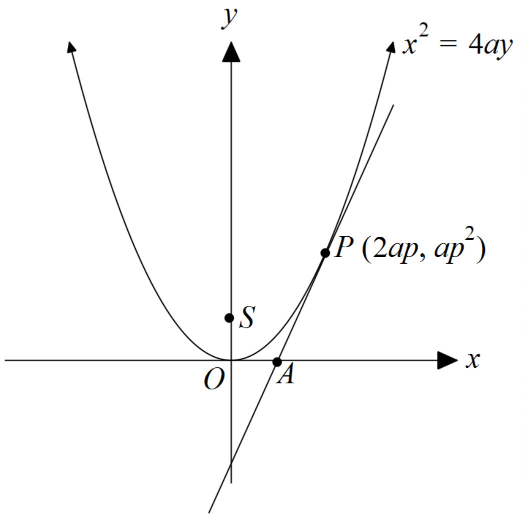
End of Question 13

Question 14 (15 Marks) Start a new writing booklet. **Marks**

a) Prove by mathematical induction that for $n \geq 2$,

$$2 \times 1 + 3 \times 2 + \ldots + n(n-1) = \frac{1}{3}\, n(n^2 - 1)$$ 3

b) A point $P\ (2ap, ap^2)$ lies on the parabola $x^2 = 4ay$. The focus of the parabola is at S. The tangent to the parabola at point P meets the x-axis at point A.



NOT TO SCALE

(i) Find the coordinates of A. **1**

(ii) Prove that $\angle PAS$ is a right angle. **1**

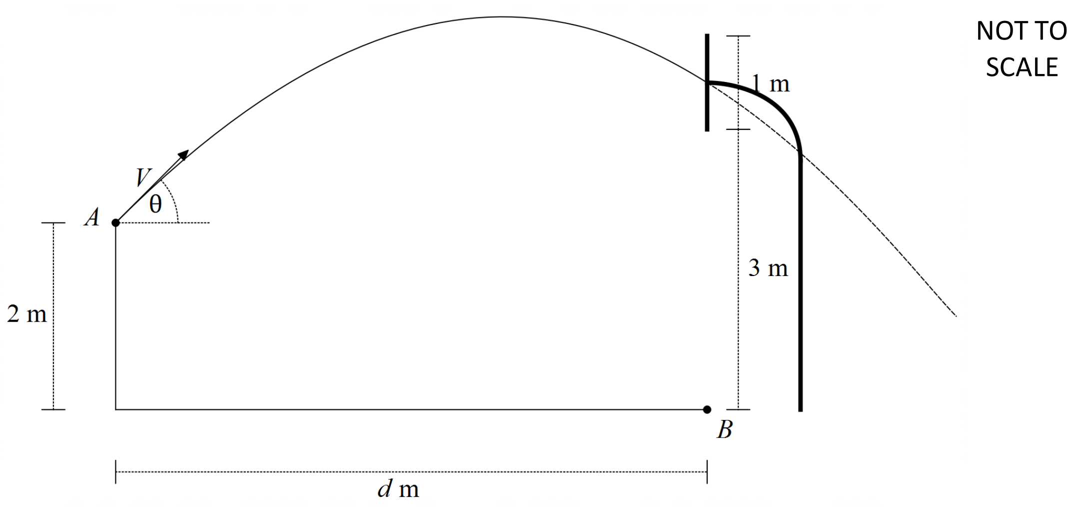
(iii) Show that the locus of the centre of the circle whose circumference passes through points P, S, and A is a parabola, giving its vertex and focal length. **3**

Question 14 continued over page

c) A basketball player throws a basketball from point A, at a height of 2 metres from the ground, with an initial velocity V and angle θ to the horizontal. The equations of motion are given by

$$x = Vt\cos\theta, \quad y = 2 + Vt\sin\theta - \frac{g}{2}t^2 \text{ (DO NOT PROVE THIS).}$$

For these questions, consider the ball to be the point at its centre, ignoring its diameter.



(i) Show that the maximum height of the ball is $h = 2 + \frac{V^2\sin^2\theta}{2g}$. **2**

The basketball is to be thrown at a distance of d metres from point B, which is on the ground directly below the bottom of the backboard. The backboard is 1 metre tall and its bottom is 3 metres above point B.

(ii) Show that the height of the ball when it reaches point B can be given by **3**

$$y = 2 - \frac{gd^2}{2V^2} + d\tan\theta - \frac{gd^2}{2V^2}\tan^2\theta$$

(iii) The player is practicing free throws by throwing the ball $d = 4.6$ metres from point B with an initial velocity of 8 m/s. The ball is thrown at an angle such that its maximum height is equal to the height of the top of the backboard. Show that the ball hits the backboard. (Use $g = 9.8\ \mathrm{m/s^2}$) **2**

End of Paper

Year 12 Mathematics Extension I Section I
Multiple Choice Answer Sheet

Student Name ______________________________

Class 12MTX ______

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample: 2 + 4 = (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

- If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ~~●~~ C ○ D ○

- If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word correct and drawing an arrow as follows.

correct

A ● B ~~●~~ C ○ D ○

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1. A ○ B ○ C ○ D ○
2. A ○ B ○ C ○ D ○
3. A ○ B ○ C ○ D ○
4. A ○ B ○ C ○ D ○
5. A ○ B ○ C ○ D ○
6. A ○ B ○ C ○ D ○
7. A ○ B ○ C ○ D ○
8. A ○ B ○ C ○ D ○
9. A ○ B ○ C ○ D ○
10. A ○ B ○ C ○ D ○

Question		Marks
MC	1-10	/10
Q11		/15
Q12		/15
Q13		/15
Q14		/15
Total		/70

MK ML RL AB KS

NAME___SOLUTIONS____________

CLASS 12MTX___

CHERRYBROOK TECHNOLOGY HIGH SCHOOL

2019

YEAR 12

AP4

MATHEMATICS EXTENSION 1

Time allowed – 2 HOURS + 5 Minutes Reading Time

DIRECTIONS TO CANDIDATES:

SECTION I: Use the multiple-choice answer sheet provided.

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- The NESA Reference Sheet is provided.

Section I

10 Marks

Attempt Questions 1 – 10.

Allow about 15 minutes for this section.

Use the multiple choice answer sheet for questions 1 – 10.

1 The acute angle between the lines $y = 2x$ and $y = 6x$ is θ.
What is the value of $\tan\theta$?

(A) $\mathbf{\frac{4}{13}}$

(B) $\frac{8}{11}$

(C) $\frac{1}{8}$

(D) $\frac{4}{9}$

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$$\tan\theta = \left|\frac{2-6}{1+2\times 6}\right| = \left|-\frac{4}{13}\right|$$
$$= \frac{4}{13}$$

2 What is the remainder when $x^3 - 10x$ is divided by $x + 5$?

(A) **–75**

(B) 75

(C) $x^2 - 2x$

(D) $x^2 - 5x + 15$

Remainder when $P(x) = x^3 - 10x$ is divided by $x + 5$

$(-5) = (-5)^3 - 10 \times (-5)$

$P = -75$

3 If $\cos 2\theta = \frac{1}{3}$ and $0 \leq \theta \leq \frac{\pi}{2}$, what is the value of $\tan\theta$?

(A) $\frac{1}{2}$

(B) $\mathbf{\frac{1}{\sqrt{2}}}$

(C) $\sqrt{2}$

(D) 2

Let $t = \tan\theta$

Then $\frac{1-t^2}{1+t^2} = \frac{1}{3}$

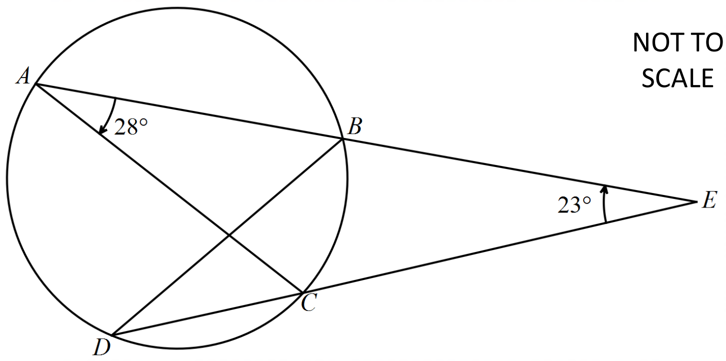
$3 - 3t^2 = 1 + t^2$

$\therefore t^2 = \frac{1}{2}$

$0 \leq \theta \leq \frac{\pi}{2} \Rightarrow t > 0$

$\therefore t = \frac{1}{\sqrt{2}}$

4 In the diagram, ABC is a circle. D is on the circumference of the circle, with AB and DC produced to meet at E. $\angle BAC = 28°$ and $\angle DEA = 23°$.



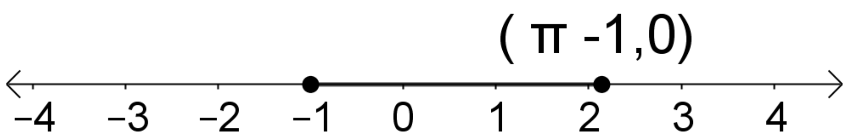
NOT TO SCALE

What is the size of $\angle DBE$?

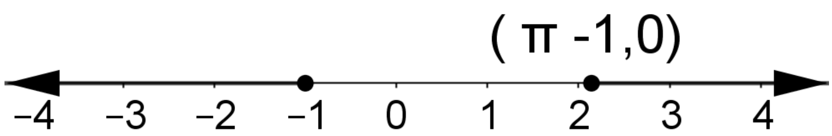
(A)	**129°**	$\angle BDC = 28°$ (angles in the same segment are equal)
(B)	134°	$\angle CBE = 180° - 28° - 23° = 129°$ (angle sum of a triangle is 180°)
(C)	152°	
(D)	141°	

5 Which diagram represents the domain of the function $f(x) = \cos^{-1}(x+1)$?

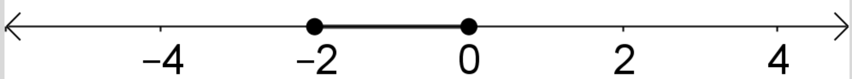
(A) $-1 \leq x + 1 \leq 1$



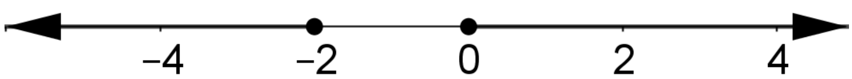
(B) $-2 \leq x \leq 0$



(C)



(D)



6 The displacement x of a particle at time t is given by $x = 4\cos 3t$

What is the maximum speed of the particle?

(A) -12

(B) 12

(C) 6

(D) -6

Taking derivative to get velocity:

$$x' = -12\sin 3t$$

Max of velocity is the amplitude of the function, 12.

7 A particle moves in a straight line. Its position at any time t is given by:

$$x = 4\sin 2t + 3\cos 2t$$

What is the acceleration in terms of x?

(A) $\boldsymbol{\ddot{x} = -4x}$

(B) $\ddot{x} = -8x$

(C) $\ddot{x} = -16x^2$

(D) $\ddot{x} = -16\sin 2x - 12\cos 2x$

$$\begin{aligned} x &= 4\sin 2t + 3\cos 2t \\ \dot{x} &= 8\cos 2t - 6\sin 2t \\ \ddot{x} &= -16\sin 2t - 12\cos 2t \\ &= -4(4\sin 2t + 3\cos 2t) \\ &= -4x \end{aligned}$$

8 What is the value of $\displaystyle\lim_{x\to 2}\frac{\sin(x-2)}{x^2+3x-10}$

(A) undefined

$$\lim_{x\to 2}\frac{\sin(x-2)}{x^2+3x-10} = \lim_{x\to 2}\frac{\sin(x-2)}{(x+5)(x-2)} = \lim_{x\to 2}\frac{\sin(x-2)}{(x-2)} \times \lim_{x\to 2}\frac{1}{x+5}$$
$$= 1 \times \frac{1}{7} = \frac{1}{7}$$

(B) 0

(C) 7

(D) $\boldsymbol{\frac{1}{7}}$

9 Which of the following is an expression for $\int \frac{x}{(2-x^2)^3}\,dx$?

Use the substitution $u = 2 - x^2$.

(A) $\int \frac{du}{u^3}$

(B) $\int \frac{du}{-u^3}$

(C) $\int \frac{du}{2u^3}$

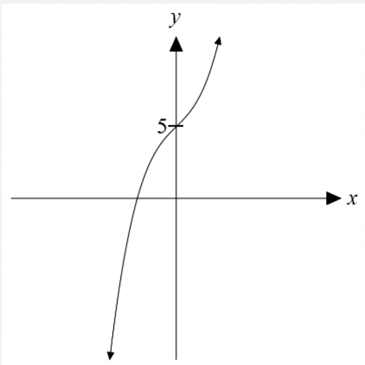
(D) $\int \frac{du}{-2u^3}$

$$u = 2 - x^2$$
$$\frac{du}{dx} = -2x$$
$$-\frac{1}{2}du = xdx$$
$$\int \frac{x}{(2-x^2)^3}\,dx = \int \frac{1}{-2u^3}\ \mathrm{du}$$

10 Consider the polynomial $p(x) = ax^3 + cx + 5$ with a and c positive.

Which graph could represent of $p(x)$?

(A)



Consider $p'(x) = 3ax^2 + c$

$p'(x) = 0$ when $x = \sqrt{-\frac{c}{3a}}$

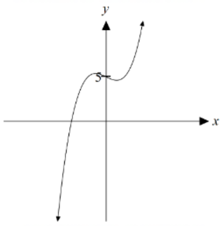
As $c > 0$ and $a > 0$, there are no real solutions. Therefore, the function has no stationary points.

If we consider $p''(x) = 6ax$, $p''(x) = 0$ when $x = 0$

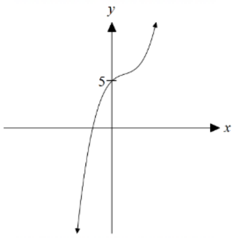
Therefore, there is a turning point at $x = 0$

The graph A shows no stationary points and a turning point at $x = 0$.

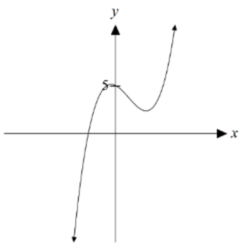
(B)



(C)



(D)



Section II

60 Marks

Attempt Questions 11 – 14.
Allow about 1 hour and 45 minutes for this section.

Answer each question in a separate writing booklet.
Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 Marks) Start a new writing booklet. **Marks**

a) The point P internally divides the interval from $A(-2,\ 2)$ to $B(3,\ 5)$ into the ratio of 3:1. Find the y-coordinate of P **1**

y-coordinate of P: $\frac{my_1+ny_2}{m+n} = \frac{3\times5+1\times2}{3+1} = \frac{17}{4}$

b) Solve $\frac{3x}{2x+3} + 1 \geq 0$. **3**

Note: $x \neq -\frac{3}{2}$

$\frac{3x}{2x+3} + 1 \geq 0$ (subtract 1 from both sides)

$\frac{3x}{2x+3} \geq -1$ (multiply both sides by $(2x+3)^2$)

$3x(2x+3) \geq -(2x+3)^2$ (expand brackets)

$6x^2 + 9x \geq -4x^2 - 12x - 9$ (move terms to LHS)

$10x^2 + 21x + 9 \geq 0$ (factorise)

$(2x+3)(5x+3) \geq 0$

Sign analysis or up-facing parabola shape reveals that the LHS is negative between the roots $x = -\frac{3}{2}$ and $x = -\frac{3}{5}$ and positive outside.

$x < -\frac{3}{2}$ or $x \geq -\frac{3}{5}$ $(x \neq -\frac{3}{2})$

3 marks: Correct answer.

2 marks: Makes some progress

1 mark: States $x \neq -\frac{3}{2}$ or multiplies by square of denominator

c) Differentiate $2x^2 \cos^{-1}(3x)$ **2**

$$4x \times \cos^{-1} 3x + 2x^2 \times \frac{-1}{\sqrt{1-9x^2}} \times 3$$

$$= 4x \cos^{-1} 3x - \frac{6x^2}{\sqrt{1-9x^2}}$$

$$= 2x\left(2\cos^{-1} 3x - \frac{3x}{\sqrt{1-9x^2}}\right)$$ (factorised form)

2 mark: Correct answer.

1 mark: Uses product rule

d) Find $\displaystyle\int \cos^2 4x\, dx$ **2**

$$\int \cos^2 4x\ dx = \int \frac{1}{2}(\cos 8x + 1)\, dx = \frac{1}{2}\left(\frac{1}{8}\sin 8x + x\right) + C$$

$= \frac{1}{16}\sin 8x + \frac{1}{2}x + C$ (expanded form)

$= \frac{1}{16}(\sin 8x + 8x) + C$ (factorised form)

$= \frac{1}{8}(4x + \sin 4x \cos 4x) + C$ (substituted $\sin 8x = 2\sin 4x\cos 4x$ form) (Any of the above forms are acceptable answers)

2 mark: Correct answer.

1 mark: Rewrites $\cos^2 x$

e) Use the substitution $u = e^{2x}$ to evaluate, **3**

$$\int_0^{\frac{1}{2}} \frac{e^{2x}}{e^{4x}+1}\,dx$$ giving your answer in exact form.

$u = e^{2x}$

$du = 2e^{2x}dx \rightarrow \frac{1}{2}du = e^{2x}dx$

When $x = \frac{1}{2}, u = e$

When $x = 0, u = 1$

$$\int_0^{\frac{1}{2}} \frac{e^{2x}}{e^{4x}+1}\,dx = \int_1^e \frac{1}{2} \times \frac{1}{u^2+1}\,du$$

$$= \frac{1}{2}[\tan^{-1} u]_1^e = \frac{1}{2}(\tan^{-1} e - \tan^{-1} 1) = \frac{1}{2}(\tan^{-1} e - \frac{\pi}{4})$$

3 marks: Correct answer.

2 marks: Rewrites integrand and integrates correctly.

1 mark: Finds the new limits and differentiates u.

f) Find the constant term of the binomial expansion of $\left(2x - \frac{3}{x^2}\right)^9$? **2**

$\left(2x - \frac{3}{x^2}\right)^9 = \sum_{k=0}^{9} \binom{9}{k} (2x)^k \left(\frac{-3}{x^2}\right)^{9-k}$ (see Reference Sheet for Binomial Thm)

$T_k = \binom{9}{k}(2x)^k \left(\frac{-3}{x^2}\right)^{9-k} = \binom{9}{k} 2^k x^k (-3)^{9-k} x^{-2(9-k)} = \binom{9}{k} 2^k (-3)^{9-k} x^{3k-18}$

Constant term occurs when power of x is 0:

$3k - 18 = 0$

$k = 6$

Coefficient:

$\binom{9}{6} 2^6 (-3)^3 = -\binom{9}{6} 2^6 3^3$

2 mark: Correct answer.

1 mark: States general term

g) Use Newton's method to find a second approximation to the root of the function $f(x) = \sin(e^x)$. Take x $=$ 5 as the first approximation. Answer correct to two decimal places. **2**

$$f(x) = \sin(e^x)$$
$$f'(x) = e^x \cos(e^x)$$

Use $x_0 = 5$

$$x_1 = x_0 - \frac{f(x)}{f'(x)}$$
$$= 5 - \frac{f(5)}{f'(5)}$$
$$= 5 - \frac{\sin(e^5)}{e^5\cos(e^5)}$$
$$= 4.9936\ldots$$
$$\approx 4.99$$

2 mark: Correct answer.

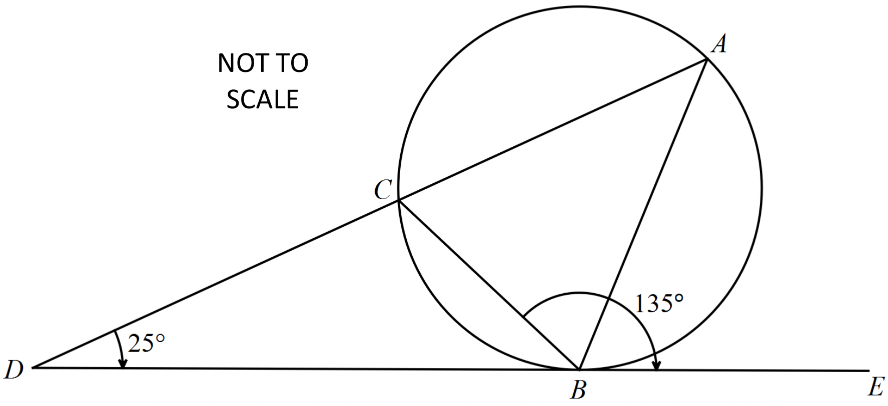
1 mark: Differentiates and finds $f'(x)$

End of Question 11

Question 12 (15 Marks) Start a new writing booklet. **Marks**

a) The points A, B, and C lie on a circle. DBE is tangent at B, and ACD is a straight line. **2**

It is given that $\angle ADE = 25°$ and $\angle CBE = 135°$.



Find the size of $\angle ABC$, giving reasons.

$\angle CBD = 45°$ (DBE is a straight line)
$\angle BAC = 45°$ (angle between tangent and a chord is equal to the angle in the alternate segment)
$\angle ACB = 25° + 45° = 70°$ (exterior angle of a triangle)
$\angle ABC = 180° - 70° - 45° = 65°$ (angle sum of a triangle)

2 mark: Correct answer.

1 mark: Shows some understanding

b) Ice cream taken out of a freezer has a temperature of -18°C. It is placed in a room with a constant temperature of 25°C. After t minutes the temperature, T°C, of the ice cream can be given by

$$T = A - Be^{-0.025t},$$

where A and B are positive constants.

(i) Show that T is a solution to $\frac{dT}{dt} = -0.025(T - A)$ **1**

Differentiating T: $\frac{dT}{dt} = -0.025(-Be^{-0.025t})$
But $-Be^{-0.025t} = T - A$, therefore: $\frac{dT}{dt} = -0.025(T - A)$

(ii) How long does it take for the ice cream to reach its melting point of -3°C? **3**

$T = A - Be^{-0.025t}$
As $t \to \infty$, $e^{-0.025t} \to 0$ and $T \to 25$ So $A = 25$
We are given that $T = -18$ when $t = 0$
$-18 = 25 - Be^{0}$ $-18 = 25 - B$ $B = 43$
So, $T = 25 - 43e^{-0.025t}$
Substitute $T = -3$:

$$-3 = 25 - 43e^{-0.025t}$$
$$-28 = -43e^{-0.025t}$$
$$\frac{28}{43} = e^{-0.025t}$$
$$\ln\frac{28}{43} = -0.025t$$

$t = \frac{\ln\frac{28}{43}}{-0.025} = 17.159 \ldots \approx 17.2$ minutes

3 marks: Correct answer.

2 marks: Substitutes $T = -3$

1 mark: Finds the values of A and B.

c) Let. $f(x) = \frac{1}{\sqrt{2x+1}}$

(i) Find the domain and range of $f(x)$. **2**

$f(x) = \frac{1}{\sqrt{2x+1}}$

2 mark: Correct answer.

Domain:

$2x + 1 \geq 0$

$x \geq -\frac{1}{2}$

1 mark: Finds either domain or range or shows some understanding

Range:

When x is close to $\frac{1}{2}$, the denominator is close to 0 and hence
As x gets closer to $\frac{1}{2}$ from the right, $y \to \infty$.
As x gets larger, the denominator gets smaller, so $y \to 0$.
$y > 0$

(ii) Find an expression for the inverse function $f^{-1}(x)$. **2**

Interchange x and y, then solve for y:

2 mark: Correct answer.

$x = \frac{1}{\sqrt{2y+1}}$ (cross multiply)

$\sqrt{2y+1} = \frac{1}{x}$ (square both sides)

$2y + 1 = \frac{1}{x^2}$ (solve for y)

$y = \frac{1}{2}\left(\frac{1}{x^2} - 1\right)$ (combine into one fraction)

$y = \frac{1-x^2}{2x^2}$

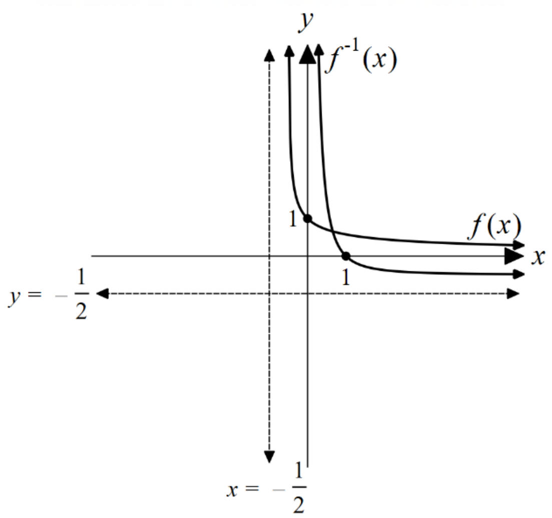
$f^{-1}(x) = \frac{1-x^2}{2x^2}$

1 mark: Interchanges x and y.

(iii) On the same set of axes, sketch the graphs $y = f(x)$ and $y = f^{-1}(x)$, clearly marking *x*- or *y*-intercepts and any asymptotes which are not the *x*- or *y*-axes. **2**

The domain of $f(x)$ is $x > -\frac{1}{2}$, and there is a vertical asymptote at $x = -\frac{1}{2}$.
The range of $f(x)$ is $y > 0$, and there is a horizonal asymptote at $y = 0$.
There is a y-intercept of 1 (found by substituting $x = 0$)
The domain and range swap for $f^{-1}(x)$.
The horizontal and vertical asymptotes also swap.
There will also be an x-intercept of 1.

2 mark: Correct sketch.

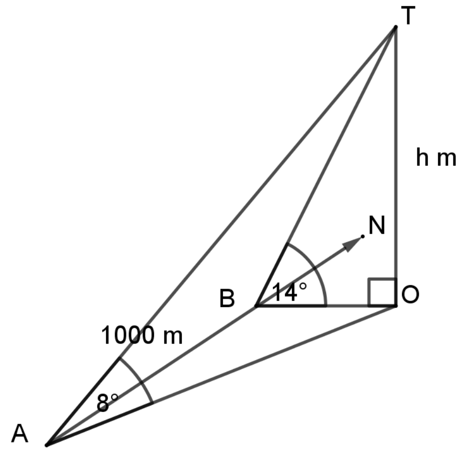


1 mark: Finds intercept and asymptotes or reflects $f(x)$ in $y = x$ or shows some understanding.

d) A person is travelling in a car along a road from point A to point B, which is 1000 metres due north of Point A. Point B is due west of a tower OT, where T is the top of the tower. The point O is directly below T and on the same horizontal plane as the road. The height of the tower above O is h metres.

From point A the angle of elevation of the top of the tower is 8°.

From point B the angle of elevation of the top of the tower is 14°.



NOT TO SCALE

(i) Show that $OB = h\cot 14°$ **1**

$\tan 14° = \dfrac{h}{OB}$

$OB = \frac{h}{\tan 14°}$

$OB = h\cot 14°$

(ii) Hence, find the value of h. **2**

Similarly to part (i), $OA = h\cot 8°$

By Pythagorean theorem:

$OB^2 + AB^2 = OA^2$ (substitute)

$(h\cot 14°)^2 + 1000^2 = (h\cot 8°)^2$ (square brackets)

$h^2\cot^2 14° + 1000^2 = h^2\cot^2 8°$ (rearrange)

$h^2\cot^2 8° - h^2\cot^2 14° = 1000^2$ (factorise h^2)

$h^2(\cot^2 8° - \cot^2 14°) = 1000^2$ (divide and square root)

$h = \frac{1000}{\sqrt{\cot^2 8° - \cot^2 14°}}$ (using $\cot\theta = \frac{1}{\tan\theta}$)

$$h = \frac{1000}{\sqrt{\left(\frac{1}{\tan 8°}\right)^2 - \left(\frac{1}{\tan 14°}\right)^2}}$$

$= 170.147\ldots \approx 170$ m

(alternatively one can use $\cot\theta = \tan(90° - \theta)$)

2 mark: Correct answer.

1 mark: Finds OA and uses Pythagoras' Theorem.

End of Question 12

Question 13 (15 Marks) Start a new writing booklet. **Marks**

a) A particle is moving along the x-axis, such that its displacement at time t is x, and its velocity is v in units of metres and seconds. The particle is moving such that its velocity is given by $v^2 = -4x^2 - 8x + 12$.

(i) Show that the particle moves in simple harmonic motion with a period of π. **2**

$v^2 = -4x^2 - 8x + 12$ (multiply both sides by $\frac{1}{2}$)

2 mark: Correct answer.

$\frac{1}{2}v^2 = -2x^2 - 4x + 6$ (differentiate both sides w.r.t. x)

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \ddot{x} = -4x - 4$ (factorise)

1 mark: Finds $\frac{1}{2}v^2$

$\ddot{x} = -4(x+1)$

This is of the form $\ddot{x} = -n^2(x-b)$ for simple harmonic motion, where $n = 2, b = -1$

The period is $\frac{2\pi}{n} = \frac{2\pi}{2} = \pi$

(ii) What is the maximum speed of the particle? **1**

The maximum speed occurs when $\ddot{x} = 0$

$-4(x+1) = 0$ when $x = -1$

Substitute into expression for v^2:

$v^2 = -4(-1)^2 - 8(-1) + 12$

$v^2 = 16$

$v = 4$

b) (i) Differentiate $x\tan^{-1}x - \frac{1}{2}\ln(1+x^2)$. **2**

$$\frac{d}{dx}\left(x\tan^{-1}x - \frac{1}{2}\ln(1+x^2)\right) = \tan^{-1}x + \frac{x}{1+x^2} - \frac{1}{2}\times\frac{2x}{1+x^2}$$

2 mark: Correct answer.

$$= \tan^{-1}x + \frac{x}{1+x^2} - \frac{x}{1+x^2} = \tan^{-1}x$$

1 mark: Uses the product rule or differentiates ln.

(ii) Hence, evaluate $\displaystyle\int_0^1 \tan^{-1}x\ dx$, giving your answer in exact form. **1**

Then $\int_0^1 \tan^{-1}x\ dx = \left[x\tan^{-1}x - \frac{1}{2}\ln(1+x^2)\right]_0^1$

$= \tan^{-1}1 - \frac{1}{2}\ln(2) - \left(0 - \frac{1}{2}\ln(1)\right)$

$= \frac{\pi}{4} - \frac{1}{2}\ln 2$

$= \frac{1}{4}(\pi - \ln 4)$ (factorised and log law applied)

c) The function $f(x)$ is given by $f(x) = \sin^{-1} x + \cos^{-1} x$

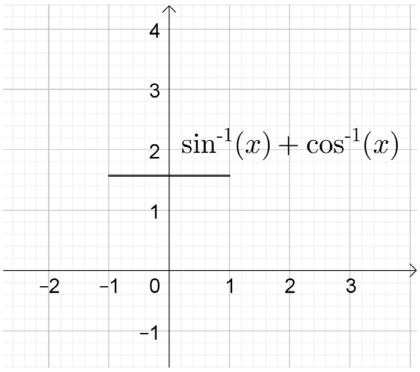
(i) Find $f'(x)$ **1**

$$f'(x) = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} = 0$$

(ii) Hence sketch the graph of $y = \sin^{-1} x + \cos^{-1} x$ **1**

$f'(x) = 0 \therefore$ horizontal line

$f(0) = \sin^{-1} 0 + \cos^{-1} 0 = 0 + \frac{\pi}{2} = \frac{\pi}{2}$



d) The velocity of a particle is given by $v = 3x + 7$ cm/s. If the particle is initially 1cm to the right of the origin, find the displacement as a function of time. **3**

$v = \frac{dx}{dt} = 3x + 7$ (taking reciprocals)

$\frac{dt}{dx} = \frac{1}{3x+7}$ (integrating)

$t = \frac{1}{3}\ln(3x + 7) + C$

When $t = 0$, $x = 1 \therefore C = -\frac{1}{3}\ln 10$

$\therefore t = \frac{1}{3}\ln(3x + 7) - \frac{1}{3}\ln 10$ (multiply by 3)

$3t + \ln 10 = \ln(3x + 7)$ (rearranging)

$3x + 7 = e^{3t + \ln 10}$ (rearranging)

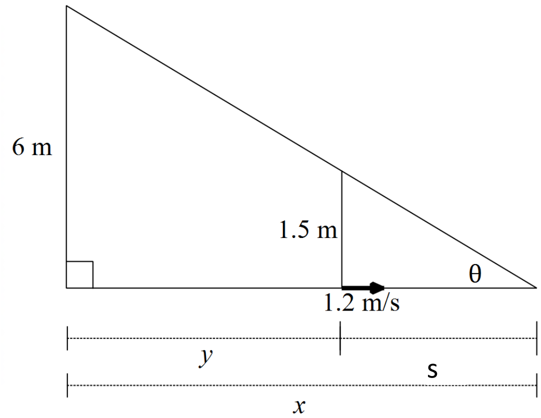
$$x = \frac{e^{3t+\ln 10} - 7}{3}$$

3 marks – correct solution

2 marks – taking reciprocal, integrating, finding constant of integration

1 mark – taking reciprocal and integrating

e) A person who is 1.5 metres tall is walking away from a light positioned on a pole that is 6 metres tall. The person walks at a constant speed of 1.2 metres per second.



(i) Using similar triangles, or otherwise, show that the rate at which s, the length of the shadow, is increasing with respect to time is 0.4 m/s. **1**

We can equate using similar triangles: $\frac{s}{1.5} = \frac{y}{4.5}$ $\qquad s = \frac{1}{3}y$

Differentiate both sides w.r.t. t: $\frac{ds}{dt} = \frac{1}{3}\frac{dy}{dt}$

Substituting $\frac{dy}{dt} = 1.2$: $\frac{ds}{dt} = \frac{1}{3} \times 1.2 = 0.4$ m/s

(ii) By differentiating $s\tan\theta = 1.5$, or otherwise, find the rate at which the angle θ is changing with respect to time when the person is 5 metres from the base of the light pole. **3**

$s\tan\theta = 1.5$

Differentiating w.r.t. t: $\frac{ds}{dt}\tan\theta + s\sec^2\theta\frac{d\theta}{dt} = 0$

Solving for $\frac{d\theta}{dt}$: $\frac{d\theta}{dt} = -\frac{\frac{ds}{dt}\tan\theta}{s\sec^2\theta}$

We need values of s and θ when $y = 5$:

$\frac{s}{1.5} = \frac{5+s}{6}$

$6s = 1.5(5+s)$

$6s = 7.5 + 1.5s$

$4.5s = 7.5$

$s = \frac{5}{3}$ m

$\tan\theta = \frac{1.5}{5/3} = 0.9$

$\theta = \tan^{-1} 0.9 \approx 42°$

Substituting (including $\frac{ds}{dt} = 0.4$):

$\frac{d\theta}{dt} = -\frac{0.4\times\tan(42°)}{\frac{5}{3}\times\sec^2(42°)} = -\frac{0.4\times\tan(42°)}{\frac{5}{3}\times\frac{1}{\cos^2(42°)}} = -0.119\ldots \approx -0.12°/\text{s}$

3 marks: Correct answer.

2 marks: Differentiates $s\tan\theta = 1.5$

1 mark: Finds the values of s and/or ϑ when $y = 5$.

Question 14 **(15 Marks)** Start a new writing booklet. **Marks**

a) Prove by mathematical induction that for $n \geq 2$,

$$2 \times 1 + 3 \times 2 + \ldots + n(n-1) = \frac{1}{3}\, n(n^2 - 1)$$ 3

Step 1: To prove the statement true for $n = 2$

$$\text{LHS} = 2 \times 1 = 2$$
$$\text{RHS} = \frac{1}{3} \times 2 \times (2^2 - 1) = 2$$

Result true for $n = 2$

Step 2: Assume the result true for $n = k$, $k \geq 2$

$$2 \times 1 + 3 \times 2 + \ldots + k(k-1) = \frac{1}{3}\, k(k^2 - 1)$$

Step 3: To prove the result true for $n = k + 1$

$$2 \times 1 + \ldots + k(k-1) + (k+1)k = \frac{1}{3}(k+1)((k+1)^2 - 1)$$

$$\begin{aligned}
\text{LHS} &= 2 \times 1 + 3 \times 2 + \ldots + k(k-1) + (k+1)k \\
&= \tfrac{1}{3}\, k(k^2 - 1) + (k+1)k \qquad \text{by assumption in step 2} \\
&= \frac{1}{3}\, [k(k+1)(k-1) + 3(k+1)k] \\
&= \frac{1}{3}\, k(k+1)[(k-1) + 3] \\
&= \frac{1}{3}\, (k+1)[k^2 + 2k] \\
&= \frac{1}{3}\, (k+1)((k+1)^2 - 1) \\
&= \text{RHS}
\end{aligned}$$

Result is true for $n = k + 1$

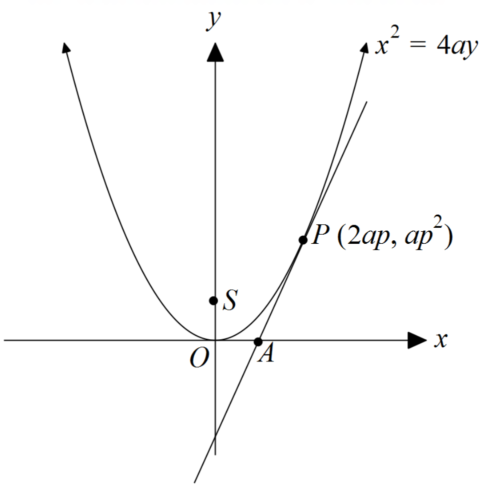
Step 4: Result true by the principle of mathematical induction.

3 marks: Correct answer.

2 marks: Proves the result true for $n = 2$ and attempts to use the result of $n = k$ to prove the result for $n = k + 1$

1 mark: Proves the result true for $n = 2$.

b) A point $P\ (2ap, ap^2)$ lies on the parabola $x^2 = 4ay$. The focus of the parabola is at S. The tangent to the parabola at point P meets the x-axis at point A.



NOT TO SCALE

(i) Find the coordinates of A. **1**

Tangent at P:

$$y = px - ap^2 \quad \text{(see Reference Sheet)}$$

A is an x-intercept, so $y = 0$:

$$0 = px - ap^2 \text{ where } p \neq 0 \qquad \text{(solve for } x\text{)}$$

$$x = ap$$

The coordinates of A are $(ap, 0)$

(ii) Prove that $\angle PAS$ is a right angle. **1**

The gradient of the tangent at p is $m_t = p = m_{PA}$

The gradient of the line through $S(0, a)$ and $A(ap, 0)$:

$$m_{AS} = \frac{0-a}{ap-0} = -\frac{1}{p}$$

$$m_t \times m_{AS} = -1$$

Therefore, $PA \perp SA$ and $\angle PAS$ is a right angle.

(iii) Show that the locus of the centre of the circle whose circumference passes through points P, S, and A is a parabola, giving its vertex and focal length. **3**

PS is the diameter of the circle through P, S, and A. (the angle in a semicircle is a right angle)

3 marks: Correct answer.

Therefore, the centre of the circle is the midpoint of PS:

$$\left(\frac{2ap}{2}, \quad \frac{ap^2 + a}{2}\right) = \left(ap, \quad \frac{ap^2 + a}{2}\right) = \left(ap, \frac{a(1 + p^2)}{2}\right)$$

2 marks: Finds Cartesian equation.

So parametrically:

$$x = ap \text{ and } y = \frac{ap^2+a}{2}$$

1 mark: Finds centre of circle

Solving x-coordinate for parameter p:

$$p = \frac{x}{a}$$

Substitute into y:

$$y = \frac{a\left(\frac{x}{a}\right)^2 + a}{2} = \frac{\frac{x^2}{a} + a}{2} = \frac{x^2 + a^2}{2a}$$

In locus form:

$$x^2 = 2ay - a^2 = 2a\left(y - \frac{a}{2}\right)$$

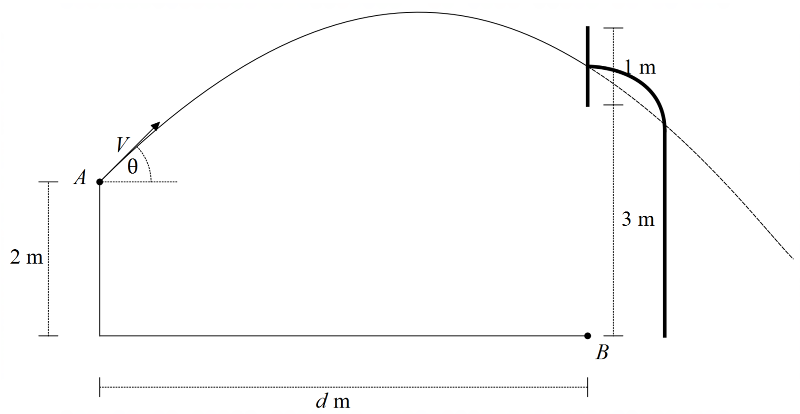
This is a parabola with vertex $\left(0, \ \frac{a}{2}\right)$ and focal length $\frac{a}{2}$.

We earlier excluded $p = 0$, as such, our locus excludes the point at $x = 0$.

c) A basketball player throws a basketball from point A, at a height of 2 metres from the ground, with an initial velocity V and angle θ to the horizontal. The equations of motion are given by

$$x = Vt\cos\theta, \quad y = 2 + Vt\sin\theta - \frac{g}{2}t^2 \text{ (DO NOT PROVE THIS).}$$

For these questions, consider the ball to be the point at its centre, ignoring its diameter.



NOT TO SCALE

(i) Show that the maximum height of the ball is $h = 2 + \frac{V^2\sin^2\theta}{2g}$. **2**

Maximum height occurs when vertical velocity is 0:

2 marks: Correct answer.

$$y' = V\sin\theta - gt = 0 \quad \text{(solve for t)}$$

$$t = \frac{V\sin\theta}{g}$$

1 mark: Finds the value of t or show some progress.

Substitute into y: $$y = 2 + V\left(\frac{V\sin\theta}{g}\right)\sin\theta - \frac{g}{2}\left(\frac{V\sin\theta}{g}\right)^2$$

$$y = 2 + \frac{V^2\sin^2\theta}{g} - \frac{V^2\sin^2\theta}{2g} = 2 + \frac{V^2\sin^2\theta}{2g}$$

Therefore, maximum height is $h = 2 + \frac{V^2\sin^2\theta}{2g}$

The basketball is to be thrown at a distance of d metres from point B, which is on the ground directly below the bottom of the backboard. The backboard is 1 metre tall and its bottom is 3 metres above point B.

(ii) Show that the height of the ball when it reaches point B can be given by $y = 2 - \frac{gd^2}{2V^2} + d\tan\theta - \frac{gd^2}{2V^2}\tan^2\theta$. **3**

Set $x = d$ $\quad x = Vt\cos\theta = d \quad$ (solve for t)

3 marks: Correct answer.

$$t = \frac{d}{V\cos\theta}$$

Substitute into y: $$y = 2 + V\left(\frac{d}{V\cos\theta}\right)\sin\theta - \frac{g}{2}\left(\frac{d}{V\cos\theta}\right)^2$$

2 marks: Converts to $\sec\theta$ and $\tan\theta$ or makes some progress.

$$y = 2 + d\frac{\sin\theta}{\cos\theta} - \frac{g}{2}\frac{d^2}{V^2\cos^2\theta} \quad \left(\text{use } \frac{1}{\cos\theta} = \sec\theta \text{ and } \frac{\sin\theta}{\cos\theta} = \tan\theta\right)$$

$$y = 2 + d\tan\theta - \frac{gd^2}{2V^2}\sec^2\theta \quad (\text{use } \sec^2\theta = 1 + \tan^2\theta)$$

$$y = 2 + d\tan\theta - \frac{gd^2}{2V^2}(1 + \tan^2\theta) \quad \text{(expand and rearrange)}$$

$$y = 2 - \frac{gd^2}{2V^2} + d\tan\theta - \frac{gd^2}{2V^2}\tan^2\theta$$

1 mark: Finds t.

(iii) The player is practicing free throws by throwing the ball $d = 4.6$ metres from point B with an initial velocity of 8 m/s. The ball is thrown at an angle such that its maximum height is equal to the height of the top of the backboard. Show that the ball hits the backboard. (Use $g = 9.8\ \text{m/s}^2$.) **2**

Find the angle using given maximum height of $h = 4$:

$$2 + \frac{8^2 \sin^2 \theta}{2 \times 9.8} = 4$$

$$\frac{8^2 \sin^2 \theta}{2 \times 9.8} = 2$$

$$\sin^2 \theta = 0.6125$$

$$\theta = \sin^{-1} \sqrt{0.6125} \approx 51.5°$$

Substitute values into equation from part (ii):

$$y_d = 2 - \frac{9.8(4.6)^2}{2(8)^2} + 4.6 \tan 51.5° - \frac{9.8(4.6)^2}{2(8)^2} \tan^2 51.5° \approx 3.6\ \text{m}$$

The ball hits the backboard as $3 < y_d < 4$.

2 marks: Correct answer.

1 marks: Finds the angle.

End of Paper